

Spinfoam models for M -theory

Harald Grosse^a

Karl-Georg Schlesinger^b

^aInstitute for Theoretical Physics

University of Vienna

Boltzmanngasse 5

A-1090 Vienna, Austria

e-mail: grosse@doppler.thp.univie.ac.at

^bErwin Schrödinger Institute for Mathematical Physics

Boltzmanngasse 9

A-1090 Vienna, Austria

e-mail: kgschles@esi.ac.at

Abstract

We use the approach to generate spin foam models by an auxiliary field theory defined on a group manifold (as recently developed in quantum gravity and quantization of BF -theories) in the context of topological quantum field theories with a 3-form field strength. Topological field theories of this kind in seven dimensions are related to the superconformal field theories which live on the worldvolumes of fivebranes in M -theory. The approach through an auxiliary field theory for spinfoams gives a topology independent formulation of such theories.

1 Introduction

In the field of quantum gravity and quantization of BF -theories there has recently been given a formulation of spinfoam models for such theories in terms of an auxiliary field theory on a group manifold which gives these theories a completely topology independent form and, hence, allows a sum over

different topologies in the path integral (see [LPR], [RR] and the literature cited therein). The simplest example (discussed in [LPR]) is BF -theory in two dimensions with an $SU(2)$ connection, i.e. we have a scalar field B , an $SU(2)$ valued 1-form A with F its 2-form field strength and the action

$$S(A, B) = \int Tr(BF)$$

In this paper, we utilize the approach through an auxiliary field theory on a group manifold to give a topology independent formulation to what we will call BH -theories, in the sequel. By this, we mean a topological field theory with a field B which is in d dimensions given by a $(d-3)$ -form (and otherwise behaves completely analogous to the B -field in BF -theory, hence the terminology) and an $SU(2)$ valued 2-form G with a 3-form field strength H with

$$H = dG$$

and the action

$$S(G, B) = \int Tr(BH)$$

Obviously, the simplest case is the three dimensional one which we will study in detail in this paper. Besides this, we will discuss how the generalization to higher dimensions, especially to seven dimensions, looks like. Topological field theories with a 3-form field strength in seven dimensions are of special interest because they are related to the six dimensional superconformal field theories living on the worldvolumes of fivebranes in M -theory (see [Dij]). This correspondence parallels the one between two dimensional superconformal and three dimensional topological field theories which is of considerable importance in perturbative string theory.

The approach through an auxiliary field theory on a group manifold allows to give a formulation of such topological field theories with a 3-form field strength which is completely independent of topology and involves a natural prescription for a sum over topologies in the path integral. The main ingredient in generalizing the approach through an auxiliary field theory from BF to BH theories will be the observation that in the discretization of a theory with a 3-form as compared to a 2-form field strength connections no longer live on edges but on two dimensional faces. This is similar to the extended lattice gauge theories introduced in [GS] (with the difference that here the colouring is taken directly from the group $SU(2)$ instead of the category of representations) and is related to the fact that 3-form theories relate in

the continuum case to gerbes on the background manifold instead of vector bundles (see [BM]).

2 The three dimensional case

We start by briefly discussing the topological content of BH theory (in complete analogy to the treatment of BF theory in [LPR]). This part is, of course, dependent on the topology of the background manifold. In the second part of this section, we will then consider the approach by an auxiliary field theory on a group manifold. By the definition of the 3-form curvature of connections on gerbes (see [BM]) it follows that a discrete approximation of the action

$$S(G, B) = \int Tr(BH)$$

is given by

$$S(B_s, U_f) = \sum_s Tr(B_s U_s)$$

where we assume that the manifold M on which we assume the theory to be defined has been approximated by a triangulation Δ and s and f are indices running over the 3-volumina and the two dimensional faces of the dual lattice Δ^* of Δ . B_s is a Lie algebra valued variable for every 3-volume s and attaching to every face f an element U_f from $SU(2)$, we define

$$U_s = U_{f_1} \dots U_{f_n}$$

for f_1, \dots, f_n the faces around the closed 3-volume s . Implicitly, this definition is dependent on a choice of ordering for the faces around a 3-volume and checking for the independence of this ordering is much more delicate than in the BF theory case (where this is trivial), see the results in [BM]. So, the main difference - as compared to the BF theory case - is, so far, the replacement of edges by faces and of faces by 3-volumina. We proceed, now, in analogy to [LPR]. Introducing the partition function

$$Z_M = \int dB_s dU_f e^{iS(B_s U_f)}$$

one verifies that

$$Z_M = \int dU_f \Pi_s \delta(U_{f_1} \dots U_{f_n})$$

by observing that the original integral involved the Fourier transform of the δ -function. Since those formulae of [LPR] which involve only $SU(2)$ representation theory remain valid for the BH theory case, of course, we can directly conclude (since this step involves basically the Peter-Weyl theorem) that

$$Z_M = \int dU_f \Pi_s \sum_j (2j+1) \text{Tr}_j(U_s)$$

and by exchanging sum and integral that

$$Z_M = \sum_{j_s} \int dU_f \Pi_s (2j_s+1) \text{Tr}_{j_s} U_s$$

where j_s is a colouring of 3-volumes by spins j and the sum is taken over all colourings.

Let us introduce additional variables v and l , running over the nodes and the edges of Δ^* , respectively. In addition let S, F, L, V be the total number of 3-volumina, faces, edges, and nodes of Δ^* , respectively. Since formula (21) of [LPR] is a formula which is only dependent on properties of $SU(2)$, again, we can conclude that adjacent faces - and therefore all faces - have to carry one and the same representation of $SU(2)$. Since each edge in the dual of a triangular lattice has three faces attached to it, we get combinatorially the same structure as in the case of BF theory with the replacement

$$\begin{aligned} f &\rightarrow s \\ l &\rightarrow f \\ v &\rightarrow l \end{aligned}$$

But while vertices are separated, this is not true for the edges, i.e. on counting the number of traces which we have to take in the foregoing formula, we have to count from the different edges attached to the same closed face only one at a time in order to avoid overcounting. In consequence, the number of traces we get is given by $L - V$ instead of L (as might be suggested by naively applying the above replacement rule). So, we get

$$Z_M = \sum_j (2j+1)^{S-F+L-V}$$

which is a topological invariant of M normally called the Euler characteristic of M in the literature. This result generalizes the fact that the partition

function of two dimensional $SU(2)$ BF theory is determined by the Euler number of the background manifold to the three dimensional case.

We now treat three dimensional BH theory in complete analogy to the spinfoam model approach to two dimensional BF theory. For an introduction to and details of the spinfoam model approach we refer the reader to [LPR].

Let Φ be a real valued scalar field on $SU(2) \times SU(2)$ which is symmetric in all its variables and has right $SU(2)$ invariance, i.e. for all $g, g_1, g_2 \in SU(2)$

$$\Phi(g_1, g_2) = \Phi(g_2, g_1)$$

and

$$\Phi(g_1, g_2) = \Phi(g_1 g, g_2 g)$$

Define an action $S(\Phi)$ by

$$\begin{aligned} S(\Phi) = & \int_{SU(2) \times SU(2)} dg_1 dg_2 \Phi^2(g_1, g_2) \\ & + \frac{\lambda}{3!} \int_{(SU(2))^3} dg_1 \dots dg_3 \Phi(g_1, g_2) \Phi(g_2, g_3) \Phi(g_3, g_1) \end{aligned}$$

It can be shown that the action $S(\Phi)$ generates through its Feynman diagrams a spinfoam model for two dimensional BF theory with $SU(2)$ connection and therefore generates quantized two dimensional BF theory with $SU(2)$ connection in a way which is not dependent on a fixed background topology and includes a natural prescription for the sum over topologies in the path integral.

The action $S(\Phi)$ consists of two parts: The kinetic term which is quadratic in the field and the interaction term. The structure of the action is determined geometrically in the following way: The number of variables of the field is given by the dimension d , the number of integrals of the interaction term is just the number of edges of the d -simplex and the number of fields in the interaction term is given by the number of nodes of the d -simplex where the edges which meet at a given node give the arguments of the fields if we assume the integration variables of the interaction term to be attached to the edges. E.g. for four dimensional BF theory, Φ is a real valued right $SU(2)$ invariant scalar field on $SU(2)^4$ which is symmetric in all its variables, the interaction term involves ten integrals and is of fifth degree in Φ .

Let us now generalize this approach to the case of $SU(2)$ BH theory in three dimensions. Let Ω be a right $SU(2)$ invariant real valued scalar

field on $SU(2) \times SU(2)$ which is symmetric. We begin by constructing the interaction term $S_I(\Omega)$. A natural guess is that the number of integrals in $S_I(\Omega)$ should be given by the number of faces of the tetrahedron while the degree in Ω should be just the number of edges of the tetrahedron with the arguments labeling the two faces meeting at each edge. So, with g_1, \dots, g_4 giving a colouring of the faces of the tetrahedron by $SU(2)$ elements, we define

$$S_I(\Omega) = \int_{SU(2)^4} dg_1 \dots dg_4 \Omega(g_1, g_2) \Omega(g_2, g_4) \Omega(g_2, g_3) \Omega(g_1, g_3) \Omega(g_1, g_4) \Omega(g_3, g_4)$$

Next, we consider the kinetic term $S_K(\Omega)$. The fact that the kinetic term of BF theory is quadratic in the field is linked to the fact that the 1-form connection A propagates along edges of Δ^* and therefore each propagator has two endpoints. For a 2-form connection the number of endpoints of the propagator is given by the number of edges around a closed face and therefore

$$S_K(\Omega) = \int_{SU(2)^2} dg_1 dg_2 \left(\sum_k \Omega^k(g_1, g_2) \right)$$

where the sum runs over the possible numbers of edges of a closed face in Δ^* . The complete action $S(\Omega)$ for three dimensional BH theory is then defined as

$$S(\Omega) = S_K(\Omega) + S_I(\Omega)$$

Comparing to the definition of connections and curvature on gerbes in [BM] one might on first sight assume that besides Ω one should introduce a field Φ on $Aut(SU(2))$ which behaves much in the way as the field in the BF theory case. While Ω corresponds to the transition elements g_{ijk} of a gerbe in [BM], the field Φ would correspond to the λ_{ij} . But observe that the λ_{ij} are not independent of the g_{ijk} and one therefore has to restrict to a single field Ω in the action in order to have the action corresponding to a properly defined configuration space.

One now checks that the action $S(\Omega)$ leads with the partition function

$$Z = \int d\Omega e^{-S(\Omega)}$$

to the amplitudes

$$A_{\Delta^*} = \int dg_e \Pi_s \delta(g_{f_1} \dots g_{f_n})$$

where f_1, \dots, f_n are, again, the faces around a closed 3-volume s and further generates three dimensional BH theory.

3 Seven dimensions

In order to generalize our findings to the higher dimensional case, we collect the following data on d -dimensional simplices: We have

$$\begin{aligned} V &= d + 1 \\ L &= \binom{d+1}{2} = \frac{d}{2} (d + 1) \\ F &= \binom{d+1}{3} = \frac{d}{6} (d + 1) (d - 1) \\ N_{F,L} &= d - 1 \end{aligned}$$

where $N_{F,L}$ denotes the number of faces that meet at an edge.

Let Δ be a triangulation of a seven dimensional manifold and Δ^* the dual, again. We define $S_K(\Omega)$ as above from Δ^* where Ω is now a right $SU(2)$ invariant real valued scalar field on $(SU(2))^6$ which is symmetric in all its variables. Then, BH theory in seven dimensions has to be generated in a topology independent way by an auxiliary field theory with the action

$$S(\Omega) = S_K(\Omega) + S_I(\Omega)$$

with

$$S_I(\Omega) = \int_{(SU(2))^6} dg_1 \dots dg_6 \Omega(\dots) \dots \Omega(\dots)$$

where $S_I(\Omega)$ is of order 28 in Ω and the arguments of the factors of Ω label the six faces meeting at an edge of the 7-simplex.

Observe that BH theory in seven dimensions includes as a special case the action

$$S(G) = \int Tr(G^2 dG)$$

for a 2-form connection G which gives the simplest possible generalization of three dimensional Chern-Simons theory to a seven dimensional theory with

a 3-form field strength. The above action $S(\Omega)$ for an auxiliary field theory which generates BH theory in seven dimensions might therefore be of interest as a simple model to begin to study properties of the seven dimensional topological field theories, arising in M -theory, in a topology independent way.

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